ASYMPTOTIC OF SOLUTION OF THE NAVIER-STOKES EQUATION NEAR THE ANGULAR POINT OF THE BOUNDARY

(ASIMPTOTIKA RESHENIIA URAVNENIIA NAV'E-STOKSA V OKRESTNOSTI UGLOVOI TOCHKI GRANITSY)

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We consider the solution of the following equation :

$$\Delta\Delta u + a\frac{\partial u}{\partial x}\frac{\partial \Delta u}{\partial y} + b\frac{\partial u}{\partial y}\frac{\partial \Delta u}{\partial x} = f \tag{1}$$

in the region G, whose boundary Γ is smooth everywhere with the exception of the origin, near which it consists of two rectilinear segments ℓ_1 and ℓ_2 of length a_0, intersecting at an angle ω $(\omega \leq 2\pi)$.

Coefficients a and b entering (1), are constants and the solution $U(x, y)$ is assumed to possess first derivatives continuous within the closed region G and becoming zero together with the normal derivative everywhere on Γ except, perhaps, at the origin.

Let us introduce some notation. Function v belongs to the space H_α^{0k}, if the integrals

$$\iint\limits_G r^{\alpha-2(k_1+k_2)}\left|\frac{\partial^{k_1+k_2}v}{\partial x^{k_1}\partial y^{k_2}}\right|^2 dxdy, \qquad k_1+k_2 \leqslant k$$

where r is the distance from the origin, are finite.

We shall denote by $\|v\|_{k\alpha}^2$ the sum of all such integrals. Function $v \in C_m$, if it has m continuous derivatives in the closed region G. By the above assumptions, the sought solution is a member of C_1 and moreover, $u \in H_{-2+\beta}^{01}$ for all $\beta > 0$.

We shall show that such a solution has, near the origin, an asymptotic of the type

$$u = \sum_{k=1}^{\infty}\sum_{j=0}^{j_k} r^{-i\lambda_k}\ln^j r\,\psi_{kj}(\varphi) \tag{2}$$

where λ_k is a set of complex numbers such, that $\operatorname{Im}\lambda_k > 1$ and ψ_{kj} are infinitely differentiable functions, while φ is a polar angle.

We shall utilize some facts known to hold for linear elliptic equations. Let us denote by S_a^p a region, the boundary of which consisits of arcs $r = a/p$, $r = ap$, and of segments $\varphi = 0$ and $\varphi = \omega$. Let the function v satisfy

$$\Delta\Delta v + \sum_{i,\,k=0}^{3} a_{ik}(x, y)\frac{\partial^k v}{\partial x^i \partial y^{k-i}} = F \tag{3}$$

where functions a_{ik} have q derivatives continuous in S_2^4

$$v(0, r) = \frac{\partial v(0, r)}{\partial\varphi} = v(\omega, r) = \frac{\partial v(\omega, r)}{\partial\varphi} = 0 \tag{4}$$

Then

$$\iint_{S_2^2}\left|\frac{\partial^{p_1+p_2}v}{\partial x^{p_1}\partial y^{p_2}}\right|^2 dxdy \leqslant c\left[\sum_{q_1+q_2=0}^{q}\iint_{S_2^4}\left|\frac{\partial^{q_1+q_2}f}{\partial x^{q_1}\partial y^{q_2}}\right|^2+|v|^2dxdy\right],\quad p_1+p_2\leqslant q+4$$

The following assertion [1] about the solution of (3) is true. If

$$a_{ik}=0,\qquad u\in H^{\circ k}_{\alpha}\qquad F\in H^{\circ k_1}_{\alpha_1}$$

$$u(0,r)=u(\omega,r)=u_\varphi(0,r)=u_\varphi(\omega,r),\qquad 0\leqslant k\leqslant k_j \tag{5}$$

then

$$u=\sum_{j=0}^{M}\sum_{k=0}^{k_j} r^{-i\lambda_j}\ln^k r\psi_{k_j}(\varphi)+u_1,\qquad u_1\in H_{\alpha_1}^{\circ k_1+4}$$

Here λ_j are the zeros of the multiplicity k_j of the function $R(\lambda)$, contained within the strip

$$1\leqslant \operatorname{Im}\lambda<k_1+3-1/2\,\alpha_1$$

Function $R(\lambda)$ is constructed in the following manner. We consider a boundary value problem

$$\Delta\Delta u=0,\qquad u(0,r)=u_\varphi(0,r)=u(\omega,r)=u_\varphi(\omega,r)=0 \tag{6}$$

in the infinite region Γ_0 $(0<\varphi<\omega)$. In polar coordinates these relations become

$$\frac{\partial}{\partial r}r\frac{\partial^2}{\partial r^2}r\frac{\partial u}{\partial r}+\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{1}{r^2}\frac{\partial^3}{\partial r\partial\varphi}r\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2}=0$$

$$u=\left.\frac{\partial u}{\partial\varphi}\right|_{\varphi=0,\ \varphi=\omega}=0 \tag{7}$$

Putting $t=\ln(1/r)$, we obtain

$$u_{tttt}+2u_{tt\varphi\varphi}+u_{\varphi\varphi\varphi\varphi}-4u_{ttt}-4u_{t\varphi\varphi}+4u_{tt}+4u_{\varphi\varphi}=0$$

Applying the Fourier transform in t, we obtain the following boundary value problem

$$u_{\varphi\varphi\varphi\varphi}-2\lambda^2u_{\varphi\varphi}+\lambda^4u+4iu-4iu_{\varphi\varphi}-4\lambda^2u+4u_{\varphi\varphi}=0,$$

$$u(0)=u'(0)=u(\omega)=u'(\omega)=0$$

where λ_j are its eigenvalues and ψ_j its eigenfunctions. Let us find all λ_j. General solution of (7) has the form

$$u=C_1\cos i\lambda\varphi+C_2\sin i\lambda\varphi+C_3\sin(i\lambda+2)\varphi+C_4\cos(i\lambda+2)\varphi,\ \lambda\neq 0,\ i,\ 2i \tag{8}$$

Conditions (6) are fulfilled if the determinant

$$R(\lambda)=\begin{vmatrix}1 & \cos i\lambda\omega & 0 & -i\lambda\sin i\lambda\omega\\ 0 & \sin i\lambda\omega & i\lambda & i\lambda\cos i\lambda\omega\\ 0 & \sin(i\lambda+2\omega) & i\lambda+2 & (i\lambda+2)\cos(i\lambda+2)\omega\\ 1 & \cos(i\lambda+2)\omega & 0 & (-i\lambda+2)\sin(i\lambda+2)\omega\end{vmatrix} \tag{9}$$

becomes equal to zero.

After some elementary transformations we obtain, from (9),

$$\begin{aligned}R(\lambda)=2\,i\lambda(i\lambda+2)-2i\lambda(i\lambda+2)\cos i\lambda\omega\cos(i\lambda+2)\omega-\\ -[(i\lambda+2)^2+(i\lambda)^2]\sin i\lambda\omega\sin(i\lambda+2)\omega=0\end{aligned} \tag{10}$$

Substitution $z=i\lambda+1$ yields for z

$$\sin^2\omega z-z^2\sin^2\omega=0 \tag{11}$$

Thus, the numbers $i(z+1)$ where z are the roots of (11), will play the part of λ_j in the expansion (4). It remains to consider the solution of (7) when $\lambda=2i$ and $\lambda=i$, i.e. when the general solution differs from (8).

If $\lambda=2i$, then the general solution of (7) has the form

$$u = C_1 \sin 2\varphi + C_2 \cos 2\varphi + C_3\varphi + C_4$$

Function u satisfies the boundary conditions, if

$$C_2 + C_4 = 0, \qquad C_1 \sin 2\omega + C_2 \cos 2\omega + C_3\omega + C_4 = 0$$
$$2C_1 + C_3 = 0, \qquad 2C_1 \cos 2\omega - 2C_2 \sin 2\omega + C_3 = 0$$

If the determinant of this system $\sin\omega\,(\sin\omega - \omega\cos\omega) \neq 0$, then only a null solution can satisfy the boundary conditions. Hence, the numbers $i(z+1)$ where z are the roots of (9) will be the indices of λ_j in expansion (4), except for $\lambda = 2i$ when $\sin\omega\,(\sin\omega - \omega\cos\omega) \neq 0$. When $\sin^2\omega = \omega \sin\omega\cos\omega$, then the number $\lambda = 2i$ is present in (4).

Let us now investigate the nonlinear equation (1). We shall have to consider a more general equation

$$L_0 u = \Delta\Delta u + a\frac{\partial u}{\partial x}\frac{\partial \Delta u}{\partial y} + b\frac{\partial u}{\partial y}\frac{\partial \Delta u}{\partial x} + \sum_{i,j=0}^{3} a_{ij}\frac{\partial^{i+j}}{\partial x^i \partial y^j}u = f \tag{12}$$

where a_{ij} are functions of the type

$$a_{ij} = \sum_{s=0}^{p}\sum_{k=0}^{k_s} r^{\mu_s}\ln^k r\, a_{skij}(\varphi), \qquad \operatorname{Re}\mu_s > i + j - 4$$

and a_{skij} are infinitely differentiable functions of the polar angle. We shall prove a number of lemmas on the solutions of (12).

Lemma 1. Let u be a solution of (12) belonging to H^{oo}_{α} and C_p, and

$$u|_\Gamma = \frac{\partial u}{\partial n}\Big|_\Gamma = 0, \qquad f \in H_\beta^{\circ s}, \qquad \beta \geqslant \alpha + 2s + 8$$

Then

$$u \in H_\beta^{\circ s+4}$$

Proof. Let us choose a number a_0 small enough to ensure that when $r < a_0$, then the boundary G will consist of rectilinear segments, and let us consider the region E_n

$$a_0/2^{n+1} \leqslant r \leqslant a_0/2^{n-1}$$

Introducing the following coordinate transformations $x = (a_0/2^n)x'$, $y = (a_0/2^n)y'$, we obtain (12) in the form

$$\Delta\Delta u + a\frac{\partial u}{\partial x'}\frac{\partial \Delta u}{\partial y'} + b\frac{\partial u}{\partial y'}\frac{\partial \Delta u}{\partial x'} + a_{ij}'\frac{\partial^{i+j}u}{\partial x'^i \partial y'^j} = f a_0^4/2^{4n}$$

Applying to u the inequality (3), we obtain

$$\iint_{E_1}\left|\frac{\partial^{p_1+p_2}u}{\partial x'^{p_1}\partial y'^{p_2}}\right|^2 dx'dy' \leqslant c\left[\iint_{E_2}\left|\frac{\partial^{q_1+q_2}f}{\partial x'^{q_1}\partial y'^{q_2}}\right|^2 a_0^8/2^{8n} + |u|^2 dxdy\right]$$

and, on returning to the previous coordinate system,

$$\iint_{E_n}\frac{a_0^{2(p_1+p_2)}}{2^{2(p_1+p_2)}}\left|\frac{\partial^{p_1+p_2}u}{\partial x^{p_1}\partial y^{p_2}}\right|^2 dxdy \leqslant c\iint_{E_{n+1}}\frac{a_0^8}{2^{8n}}\left|\frac{\partial^{q_1+q_2}f}{\partial x^{q_1}\partial y^{q_2}}\right|^2 + |u|^2 dxdy$$

Summation of these inequalities yields the final result

$$\iint_G\left|\frac{\partial^{p_1+p_2}u}{\partial x^{p_1}\partial y^{p_2}}\right|^2 r^{\alpha+2p_1+2p_2}dxdy \leqslant c\iint_G\left[r^{\alpha+2p_1+2p_2+8-2q_1-2q_2}\left|\frac{\partial^{q_1+q_2}f}{\partial x^{q_1}\partial y^{q_2}}\right|^2 + r^\alpha|u|^2\right]dxdy$$

Lemma 2. Equation $\Delta\Delta u = r^\beta \ln^s r\,\Phi_{\beta s}(\varphi)$ has a particular solution of the form

$$u_1 = \sum_{j=0}^{p+s} r^{\beta+4}\ln^j r\,\Phi_j(\varphi)$$

satisfying the boundary conditions

$$u_1(0, r) = u_1(\omega, r) = \frac{\partial}{\partial \varphi} u_1(0, r) = \frac{\partial}{\partial \varphi} u_1(\omega, r) = 0$$

Number p is equal to the multiplicity of the root $\beta + 1$ in Equation (11). Existence of such a particular equation can be verified directly by substitution. Similar proof is given in [2].

Lemma 3. Let $Z = r^{\lambda} \ln^k r\psi(\varphi)$ be an arbitrary function infinitely differentiable in φ, and let $H > 0$ be any number. There exists a function v of the form

$$v = \sum_{j=0}^{p} \sum_{k=0}^{k_j} r^{\lambda_j} \ln^k r \psi_{kj}(\varphi), \qquad \operatorname{Re} \lambda_j \geqslant 4$$

satisfying the conditions

$$v(0, r) = v(\omega, r) = \frac{\partial}{\partial \varphi} v(0, r) = \frac{\partial}{\partial \varphi} v(\omega, r) = 0, L_0(v) - Z = o(r^H)$$

Proof. We shall seek function v in the form $v = v_1 + w_1$.

Taking the solution of Equation $\Delta\Delta v_1 = Z_1$ which is by Lemma 2 exists, as v_1 and making in (12) the substitution $v - v_1 = w_1$, we obtain, for w_1

$$L_1 w_1 = \sum_{j=1}^{p} \sum_{k=0}^{k_j} r^{\lambda_j} \ln^k r \psi_{kj}^{(1)}(\varphi)$$

Here L_1 is an operator similar to L_0. Let us then assume that $w_1 = v_2 + w_2$ where w_2 is the solution of $\Delta\Delta v_2 = Z_1$ satisfying the requirements of Lemma 2. We obtain, for the function w_2, Equation $L_2 w_2 = Z_2$

$$z_2 = \sum_{j=1}^{p} \sum_{k=0}^{k_j} r^{\lambda_j} \ln^k r \psi_{kj}^{(2)}(\varphi), \qquad \operatorname{Re} \lambda_j \geqslant \operatorname{Re} \lambda + 2$$

Repeating the above process, we can establish after a finite number of steps that the function $v = v_1 + v_2 + \dots + v_n$ satisfies all the requirements of Lemma 3. We shall show the validity of (2) for the solution of (1). Consider the function $u_1 = \theta u$ where θ is an infinitely differentiable function equal to zero everywhere except that vicinity of the coordinate origin, in which the boundary of G consists of rectilinear segments, and equal to unity in some vicinity of the origin (e. g. when $r \leq a_0/2$). Function u_1 satisfies Equation

$$\Delta\Delta u_1 + a \frac{\partial u_1}{\partial x} \frac{\partial \Delta u_1}{\partial y} + b \frac{\partial u_1}{\partial y} \frac{\partial \Delta u_1}{\partial x} = f_1 \qquad (13)$$

where $f_1 \equiv f$ in some vicinity of the origin. Let us represent f_1 as

$$f_1 = P_1(x, y) + F_1, \qquad F_1 = o(r^m)$$

Here P_1 is an mth degree polynomial. By Lemma 3, there exists a function

$$v = \sum_{j=1}^{p} \sum_{k=0}^{k_j} r^{i\lambda_j} \ln^k r \psi_{kp}(\varphi)$$

such, that

$$Lv - P_1 = o(r^m), \quad v(0, r) = v(\omega, r) = \frac{\partial v(0, r)}{\partial \varphi} = \frac{\partial v(\omega, r)}{\partial \varphi} = 0$$

Let us make a substitution $u_1 = v + Z$ in (11). Then, for Z we obtain

$$L_1 Z = F_2, \qquad F_2 = o(r^m), \qquad F_2 \in \overset{\circ}{H}{}^{m}_{\beta}, \ \beta > -2 \qquad (14)$$

Function Z belongs to C_1 and to $\overset{\circ\circ}{H}_{\alpha-6}$ when $\alpha > 0$. Consequently, by Lemma 1

$$z \in \overset{\circ}{H}{}^{m}_{\alpha+2m-6}$$

Let us write (14) as

$$\Delta\Delta z = \Phi_2, \qquad \Phi_2 \in H^{\circ m-3}_{-6+\alpha+2m+\delta_1}$$

$$z = \sum_{k=1}^{p}\sum_{j=0}^{j_k} r^{\lambda_k} \ln^j r \psi_{kj}^{(0)}(\varphi) + z_1, \qquad z_1 \in H^{\circ m+1}_{-4+\alpha+2m}, \qquad \operatorname{Re}\lambda_k \geqslant 2$$

By the inclusion theorem we have $Z_1 \in C_2$. Function Z_1 satisfies Equation

$$L_1 z_1 = P_3 + g_3, \qquad P_3 \in H^{\circ m-2}_{-4+\alpha+2m+\delta_1}, \quad g_3 = \sum_{k=1}^{p}\sum_{j=0}^{j_k} r^{\lambda_k} \ln^j r g_{kj}(\varphi), \qquad \operatorname{Re}\lambda_k > -1$$

Let us now find, according to Lemma 3 , function v_1 such, that

$$L_1 v_1 - g_3 \in H^{\circ m-2}_{-6+\alpha+2m+\delta_1}$$

Substituting $u_2 = z_2 + v_2$, we obtain

$$L_1 z_2 = \psi_2, \qquad \psi_2 \in H^{\circ m-2}_{-6+\alpha+2m+\delta_1}$$

which again can be written as

$$\Delta\Delta z_2 = F_2^{(1)}, \qquad F_2^{(1)} \in H^{\circ m-2}_{-6+\alpha+2m+\delta_1}$$

and, on applying (4), yield

$$z_2 = \sum_{j=1}^{p}\sum_{k=0}^{k_j} r^{\lambda_j} \ln^k r \psi_{kp}^{(1)}(\varphi) + z_3$$

Continuing this process, we can find further terms of the asymptotic and the solution $u(x, y)$ can therefore, be represented as

$$u(x, y) = \sum_{j=1}^{p}\sum_{k=0}^{k_j} r^{\gamma_j} \ln^k r \psi_{kp}(\varphi) + w \qquad w \in H^{0m+4}_{\beta}, \quad \beta > -1, \quad \gamma_k = i(\mu_k + n)$$

where μ_k are the roots of (11) . We should note that the first term of the asymptotic (2) is defined by the principal part of Equation (1) only. The exponent γ_1 is, in this case, the number $i\mu_1$ (where μ_1 is that of the roots of (11) lying above the straight line $\operatorname{Im}\lambda = 2$, which has a smallest imaginary part).

As an example, let us consider the case $\omega = 2\pi$. Here we have $\mu_k = n/2$ and we obtain the following expression for u

$$u = \sum_{n \geqslant 3}\sum_{k=0}^{k_n} r^{n/2} \ln^k r \psi_{nk}(\varphi) \qquad (n = 3, 4, \ldots,; \ k_3 = 0, \ k_4 = 1)$$

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